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Integral transforms of functions to be in certain class defined by the combination of starlike and convex functions

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ABSTRACT

Let $P_\gamma(\beta)$, $\beta < 1$, denote the class of all normalized analytic functions f in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\mathrm{Re}\left(e^{i\phi}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta\right)\right) > 0, \quad z \in \mathbb{D}$$

for some $\phi \in \mathbb{R}$. Let $M(\mu, \alpha)$, $0 \le \mu < 1$, denote the Pascu class of α-convex functions of order μ and given by the analytic condition

$$\mathrm{Re}\,\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} > \mu$$

which unifies $S^*(\mu)$ and $C(\mu)$, respectively, the classes of analytic functions that map $\mathbb{D}$ onto the starlike and convex domain. In this work, we consider integral transforms of the form

$$V_\lambda(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt.$$

The aim of this paper is to find conditions on $\lambda(t)$ so that the above transformation carry $P_\gamma(\beta)$ into $M(\mu, \alpha)$. As applications, for specific values of $\lambda(t)$, it is found that several known integral operators carry $P_\gamma(\beta)$ into $M(\mu, \alpha)$.

1. Introduction and key lemmas

Let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. A function $f \in \mathcal{S}$ is said to be starlike or convex, if f maps $\mathbb{D}$ conformally onto the domains, respectively, starlike with respect to origin and convex. Note that f is convex in $\mathbb{D}$ if and only if zf' is starlike in $\mathbb{D}$ follows from the well-known Alexander theorem (see [1] for details).

The generalization of these two classes are given by the following analytic characterizations;

$$S^*(\mu) := \left\{f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \mu,\ 0 \le \mu < 1\right\}$$

$$K(\mu) := \left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \mu,\ 0 \le \mu < 1\right\},$$

so that $S^*(0) \equiv S^*$ and $K(\mu) \equiv K$ are the starlike and convex classes respectively.

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A function $f \in \mathcal{A}$ is said to be in the Pascu class of α-convex functions of order μ $(0 \leq \mu < 1)$ if [2]

$$\mathrm{Re}\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} > \mu.$$

or in other words

$$\alpha zf'(z) + (1-\alpha)f(z) \in \mathcal{S}^*(\mu).$$

This class is denoted by $M(\alpha, \mu)$. Note that $M(0, \mu) = S^*(\mu)$ and $M(1, \mu) = K(\mu)$ which implies that $M(\alpha, \mu)$ is a smooth passage between the class of starlike and convex functions.

Further, $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{D}$, with respect to a starlike function g, if f satisfies the analytic characterization, $\mathrm{Re}\left(e^{i\sigma}\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{D},\ \sigma \in \mathbb{R}$. These close-to-convex functions f satisfy a nice geometric property that the complement of image of $\mathbb{D}$ under f are the union of closed halflines such that the corresponding open halflines are disjoint [3, Theorem 2.12, p. 52]. We denote by $\mathcal{C}$ the class of all close-to-convex functions in $\mathbb{D}$.

The main objective of this work is to find conditions on the non-negative real valued integrable function $\lambda(t)$ satisfying $\int_0^1 \lambda(t)dt = 1$, such that the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}dt \tag{1.1}$$

is in the class $M(\alpha, \mu)$. Note that this operator was introduced in [4]. To investigate this admissibility property the class to which the function f belongs is important. Let $P_\gamma(\beta)$, $\beta < 1$, denote the class of all normalized analytic functions f in the unit disc $\mathbb{D}$ such that

$$\mathrm{Re}\left(e^{i\phi}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta\right)\right) > 0, \quad z \in \mathbb{D}$$

for some $\phi \in \mathbb{R}$. This class and its particular cases were considered by many authors to prove that the operator given by (1.1) is univalent under certain conditions and in $M(\alpha, \mu)$ for some particular values of α and μ. This work was motivated in [4] by studying the conditions under which $V_\lambda(P_1(\beta)) \subset M(0, 0)$ and generalized in [5] by studying the case $V_\lambda(P_\gamma(\beta)) \subset M(0, 0)$. In [6] the conditions under which $V_\lambda(P_1(\beta)) \subset M(1, 0)$ were studied. An extensive study of $V_\lambda(P_\gamma(\beta))$ to the class $M(0, \mu)$ is in [7] and to the class $M(1, \mu)$ is in [8].

One of the main tools in the objective of this work is the following. If f and g are in $\mathcal{A}$ with the power series expansions $f(z) = \sum_{k=0}^\infty a_k z^k$ and $g(z) = \sum_{k=0}^\infty b_k z^k$ respectively, then the convolution or Hadamard product of f and g is given by $h(z) = \sum_{k=0}^\infty a_k b_k z^k$.

For $\Lambda : [0, 1] \to \mathbb{R}$ integrable over $[0, 1]$ and positive on $(0, 1)$, let

$$L_\Lambda(f) := \inf_{z\in\Delta}\int_0^1 \Lambda(t)\left(\mathrm{Re}\frac{f(tz)}{tz} - \frac{1}{(1+t)^2}\right)dt, \quad f \in \mathcal{C}, \tag{1.2}$$

and

$$L_\Lambda(\mathcal{C}) = \inf_{f\in\mathcal{C}} L_\Lambda(f). \tag{1.3}$$

Fournier and Ruscheweyh [4] have established the following:

Theorem 1.1. (i) *If $\frac{\Lambda(t)}{1-t^2}$ is decreasing on $(0, 1)$ then $L_\Lambda(\mathcal{C}) = 0$.*

(ii) *If $\lambda : [0, 1] \to \mathrm{Re}\,R$ is non-negative with $\int_0^1 \lambda(t)dt = 1$, $\Lambda(t) = \int_t^1 \lambda(t)\frac{dt}{t}$ satisfies $t\Lambda(t) \to 0$ for $t \to 0+$ and*

$$\mathcal{V}_\lambda(f) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \quad f \in \mathcal{A}, \tag{1.4}$$

then for $\beta_\lambda < 1$ given by

$$\frac{\beta_\lambda}{1-\beta\lambda} = -\int_0^1 \lambda(t)\frac{1-t}{1+t}dt, \tag{1.5}$$

we have for $\beta = \beta_\lambda : \mathcal{V}_\lambda(\mathcal{P}_\beta) \subset \mathcal{S}$ and

$$\mathcal{V}_\lambda(\mathcal{P}_\beta) \subset \mathcal{S}^* \Leftrightarrow L_\Lambda\mathcal{C} = 0.$$

For $\beta < \beta_\lambda$ there exists $f \in \mathcal{P}_\beta$ with $\mathcal{V}_\lambda(f)$ not even locally univalent.

Ali and Singh [6] proved a similar result for the functional $M_\Lambda(\mathcal{C})$ defined as $M_\Lambda(\mathcal{C}) = \inf_{f\in\mathcal{C}} M_\Lambda(f)$ where

$$M_\Lambda(f) = \inf_{z\in\Delta} \int_0^1 \Lambda(t) \left[\operatorname{Re}\left(f'(zt) - \frac{1-t}{(1+t)^3}\right)\right] dt. \tag{1.6}$$

It is clear that duality theory and techniques of [4] will be applicable if in (1.2) one replaces $\frac{f(tz)}{tz}$ by some other linear functional. The following results are generalization of the above mentioned results.

Theorem 1.2 (*[7]*). *Let $f \in \mathcal{P}_\gamma(\beta)$, $\gamma > 0$, $0 \le \mu \le 1/2$ and $\beta < 1$ with*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g_\gamma^\mu(t) dt,$$

where $g_\gamma^\mu(t)$ is given by

$$1 + g_\gamma^\mu(t) = \frac{2}{\gamma(1-\mu)} t^{-1/\gamma} \int_0^1 u^{1/\gamma-1} \frac{1-\mu(1+u)}{(1+u)^2} du. \tag{1.7}$$

Define $\Lambda_\gamma(t) = \int_t^1 \frac{\lambda(s)}{s^{1/\gamma}} ds$. Assume that $\lim_{t\to 0+} t^{1/\gamma}\Lambda_\gamma(t) = 0$ Then

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt \tag{1.8}$$

is in $M(0,\mu) = \mathcal{S}^*(\mu)$, if and only if

$$\operatorname{Re} \int_0^1 t^{1/\gamma-1} \Lambda_\gamma(t) \left[\frac{h(zt)}{tz} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right] dt \ge 0, \quad z \in \mathbb{D}, \tag{1.9}$$

where

$$h(z) =: h_\mu(z) = z\left(1 + \frac{\epsilon + 2\mu - 1}{2(1-\mu)} z\right)(1-z)^{-2}, \quad |\epsilon| = 1. \tag{1.10}$$

Theorem 1.3 (*[7]*). *Let $\mu \in [0, 1/2]$. Assume Λ_γ is integrable on $[0, 1]$ and positive on $(0, 1)$. Assume further that*

$$\Lambda_\gamma(t) \left(\log\left(\frac{1}{t}\right)\right)^{-1-2\mu}$$

is decreasing on $(0, 1)$. Then for $1/2 \le \gamma \le 1$, $L_{\Lambda_\gamma}(h) \ge 0$, where h is defined by (1.10) and

$$L_{\Lambda_\gamma}(h) = \inf_{z\in\mathbb{D}} \int_0^1 t^{1/\gamma-1} \Lambda_\gamma(t) \left[\operatorname{Re} \frac{h(zt)}{tz} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right] dt.$$

Similarly for the convex class $M(1,\mu)$ the following results were obtained in [8]

Theorem 1.4 (*[8]*). *Let $\gamma \in [1/2, 1]$ and $\Lambda_\gamma(t)$ be an integrable function on $[0, 1]$ defined by $\Lambda_\gamma(t) = \int_t^1 \frac{\lambda(s)}{s^{1/\gamma}} ds$ such that*

$$\lim_{t\to 0+} t^{1/\gamma} \Lambda_\gamma(t) = 0.$$

Suppose that $0 \le \mu \le 1/2$,

$$\frac{(1-(1/\gamma))\Lambda_\gamma(t) - t\Lambda'_\gamma(t)}{(\log(1/t))^{1+2\mu}} = \frac{(1-(1/\gamma))\Lambda_\gamma(t) + t^{1-(1/\gamma)}\lambda(t)}{(\log(1/t))^{1+2\mu}} \tag{1.11}$$

is decreasing on $(0, 1)$ and $\lambda(t)$ is decreasing on $[0, 1]$. Then we have

(i) $M_{\Lambda_\gamma}(h_\mu) \ge 0$
(ii) $V_\lambda(P_\gamma(\beta)) \subset K(\mu)$ *if and only if* $M_{\Lambda_\gamma}(h_\mu) \ge 0$, *where $\beta < 1$ is given by*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\int_0^1 \lambda(t) q_\gamma^\mu(t) dt. \tag{1.12}$$

Here $q^{\mu}_{\gamma}(t)$ *and* $M_{\Lambda_\gamma}(h_\mu)$ *are defined, respectively, as*

$$q^{\mu}_{\gamma}(t) = \frac{1}{\gamma(1-\mu)} t^{-1/\gamma} \int_0^t u^{(1/\gamma)-1} \frac{(1-\mu)-(1+\mu)u}{(1+u)^3} du \tag{1.13}$$

and

$$M_{\Lambda_\gamma}(h_\mu) = \inf_{z\in\mathbb{D}} \int_0^t t^{(1/\gamma)-1} \Lambda_\gamma(t) \left[\operatorname{Re} h'_\mu(zt) - \frac{(1-\mu)-(1+\mu)t}{(1+t)^3}\right] dt. \tag{1.14}$$

2. Main results

Based on the results given in Theorems 1.2–1.4 we can define the functional appropriately so that the integral transforms of the form (1.8) carry functions from $P_\gamma(\beta)$ into the class $M(\alpha,\mu)$, $\mu \in [0, 1/2]$. We need to introduce some basic notation. Let $q^{\mu}_{\gamma}(t)$ be the solution of the initial value problem

$$\frac{d}{dt}(t^{1/\gamma} q^{\mu}_{\gamma}(t)) = \frac{1}{\gamma} \frac{t^{(1/\gamma)-1}[(1-\mu)-(1+\mu)t]}{(1-\mu)(1+t)^3}, \quad q^{\mu}_{\gamma}(0) = 1.$$

Solving the above initial problem, we get the solution as (1.13). For the function $\Lambda_\gamma(t)$, we define

$$N_{\Lambda_\gamma}(h_\mu) = \inf_{z\in\Delta} \int_0^1 t^{\frac{1}{\gamma}-1} \Lambda_\gamma(t) h_{\mu,\alpha,z}(t) dt, \tag{2.1}$$

where

$$h_{\mu,\alpha,z}(t) = (1-\alpha)\left(\operatorname{Re}\frac{h_\mu(zt)}{zt} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right) + \alpha\left(\operatorname{Re} h'_\mu(zt) - \frac{(1-\mu)-(1+\mu)t}{(1+t)^3}\right).$$

We prove the main result.

Theorem 2.1. *Let* $\mu \in [0, 1/2]$, $\gamma \in [1/2, 1]$ *and* $\Lambda_\gamma(t)$ *be an integrable function on* $[0, 1]$ *defined by* $\Lambda_\gamma(t) = \int_t^1 \frac{\lambda(s)}{s^{1/\gamma}}$ *such that*

$$\lim_{t\to 0+} t^{1/\gamma}\Lambda_\gamma(t) = 0.$$

Suppose that for $0 \le \mu \le 1/2$,

$$\frac{\alpha t^{(1/\alpha)-(1/\gamma)-1} d\left(t^{(1/\gamma)-(1/\alpha)}\Lambda_\gamma(t)\right)}{\left(\log\frac{1}{t}\right)^{1+2\mu}} \tag{2.2}$$

is increasing where $d(.)$ *denotes the derivative with respect to* t. *Then we have*

(i) $N_{\Lambda_\gamma}(h_\mu) \ge 0$.
(ii) $V_\lambda(P_\gamma(\beta)) \subset M(\mu,\alpha)$ *if and only if* $N_{\Lambda_\gamma}(h_\mu) \ge 0$ *where* $\beta < 1$ *is given by* (1.12).

Proof.

$$\begin{aligned} I &= \int_0^1 t^{\frac{1}{\gamma}-1}\Lambda_\gamma(t)\left[(1-\alpha)\left(\operatorname{Re}\frac{h_\mu(zt)}{zt} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right) + \alpha\left(\operatorname{Re} h'_\mu(zt) - \frac{(1-\mu)-(1+\mu)t}{(1+t)^3}\right)\right] dt \\ &= \int_0^1 t^{\frac{1}{\gamma}-1}\Lambda_\gamma(t)\left[(1-\alpha)\left(\operatorname{Re}\frac{h_\mu(zt)}{zt} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right) + \alpha\frac{d}{dt}\left(\operatorname{Re}\frac{h_\mu(zt)}{z} - \frac{t(1-\mu(1+t))}{(1-\mu)(1+t)^2}\right)\right] dt \\ &= \int_0^1 t^{\frac{1}{\gamma}-1}\Lambda_\gamma(t)(1-\alpha)\left(\operatorname{Re}\frac{h_\mu(zt)}{zt} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right) dt \\ &\quad + \alpha\int_0^1 t^{\frac{1}{\gamma}-1}\left[\left(1-\frac{1}{\gamma}\right)\Lambda_\gamma(t) - t\Lambda'_\gamma(t)\right]\left(\operatorname{Re}\frac{h_\mu(zt)}{zt} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right) dt. \end{aligned}$$

With our hypothesis and from Theorem 1.2, the first integral is positive and from Theorem 1.4 and our hypothesis the second integral also positive and hence the first part of the theorem is proved. For the second part, let

$$F(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t} dt.$$

Then

$$F'(z) = \int_0^1 \lambda(t) f'(tz)dt = \int_0^1 \lambda(t)\frac{1}{(1-tz)}dt * f'(z).$$

Since

$$F \in M(\mu,\alpha) \Longleftrightarrow \alpha z F' + (1-\alpha)F \in \mathcal{S}^*(\alpha)$$

it is enough to check the condition with respect to starlikeness. From [9, p. 94], we have

$$g \in \mathcal{S}^*(\alpha) \Longleftrightarrow \frac{1}{z}(F * h)(z) \neq 0, \quad z \in \mathbb{D},$$

where $h(z)$ is given by (1.10). This means

$$\begin{aligned}
0 \neq\ & \frac{\alpha z F'(z) + (1-\alpha)F(z) * h_\mu(z)}{z} \\
&= (1-\alpha)\int_0^1 \lambda(t)\frac{dt}{1-tz} * \frac{h_\mu(z)}{z} * \frac{f(z)}{z} + \alpha \int_0^1 \lambda(t)\frac{dt}{1-tz} * h'_\mu(z) * \frac{f(z)}{z} \\
&= (1-\alpha)\int_0^1 \lambda(t)\frac{dt}{1-tz} * \left(\frac{1-\beta}{\gamma z^{1/\gamma}}\int_0^z w^{1/\gamma-1}\frac{1+xz}{1+yz}dw + \beta\right) * \frac{h_\mu(z)}{z} \\
&\quad + \alpha \int_0^1 \lambda(t)\frac{dt}{1-tz} * \left(\frac{1-\beta}{\gamma z^{1/\gamma}}\int_0^z w^{1/\gamma-1}\frac{1+xz}{1+yz}dw + \beta\right) * h'_\mu(z) \\
&= (1-\beta)(1-\alpha)\int_0^1 \lambda(t)\left(\frac{1}{\gamma z^{1/\gamma}}\int_0^z w^{1/\gamma-1}\frac{h_\mu(tw)}{tw}dw + \frac{\beta}{1-\beta}\right)dt * \frac{1+xz}{1+yz} \\
&\quad + \alpha(1-\beta)\int_0^1 \lambda(t)\left(\frac{1}{\gamma z^{1/\gamma}}\int_0^z w^{1/\gamma-1}h'_\mu(tw)dw + \frac{\beta}{1-\beta}\right)dt * \frac{1+xz}{1+yz},
\end{aligned}$$

which clearly holds if and only if

$$\begin{aligned}
\mathrm{Re}\ \Bigg(&(1-\beta)(1-\alpha)\int_0^1 \lambda(t)\left(\frac{1}{\gamma z^{1/\gamma}}\int_0^z w^{1/\gamma-1}\frac{h_\mu(tw)}{tw}dw + \frac{\beta}{1-\beta}\right)dt \\
&+ \alpha(1-\beta)\int_0^1 \lambda(t)\left(\frac{1}{\gamma z^{1/\gamma}}\int_0^z w^{1/\gamma-1}h'_\mu(tw)dw + \frac{\beta}{1-\beta}\right)dt\Bigg) > \frac{1}{2}
\end{aligned}$$

or equivalently

$$\begin{aligned}
\mathrm{Re}\ \Bigg(&(1-\alpha)\int_0^1 \lambda(t)\left(\frac{1}{\gamma z^{1/\gamma}}\int_0^z w^{1/\gamma-1}\frac{h_\mu(tw)}{tw}dw + \frac{\beta-(1/2)}{1-\beta}\right)dt \\
&+ \alpha\int_0^1 \lambda(t)\left(\frac{1}{\gamma z^{1/\gamma}}\int_0^z w^{1/\gamma-1}h'_\mu(tw)dw + \frac{\beta-(1/2)}{1-\beta}\right)dt\Bigg) > 0.
\end{aligned}$$

Substituting $\frac{\beta-(1/2)}{1-\beta}$ from (1.12) and simplifying gives

$$\begin{aligned}
(1-\alpha)&\int_0^1 t^{1/\gamma-1}\Lambda_\gamma(t)\left(\mathrm{Re}\,\frac{h_\mu(tz)}{tz} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right)dt \\
&+ \alpha\int_0^1 t^{1/\gamma-1}\Lambda_\gamma(t)\left(\mathrm{Re}\, h'_\mu(tz) - \frac{1-\mu-(1+\mu)t}{(1-\mu)(1+t)^3}\right)dt \geq 0
\end{aligned}$$

which means that $N_{\Lambda_\gamma}(h_\mu) \geq 0$. This completes the proof of part (ii). □

3. Applications

For our applications, we need some preparation and then we apply Theorem 2.1. In order to apply Theorem 2.1 with $\mu \in [0, 1/2]$, the main part is to show that the function g defined by

$$g(t) = \frac{\alpha t^{1-1/\gamma}\lambda(t) + (1-\alpha/\gamma)\Lambda_\gamma(t)}{(\log(1/t))^{1+2\mu}}$$

is decreasing on (0, 1), where $\Lambda_\gamma(t)$ is given by (1.11). Note that in our applications each $\lambda(t)$ that we have chosen in this section satisfies the condition $\lambda(1) = 0$ and hence, in our discussion here we assume that this condition hold. Using the fact that

$$\Lambda'_\gamma(t) = -\frac{\lambda(t)}{t^{1/\gamma}},$$

it can be easily seen that

$$g'(t) = \frac{\psi(t)}{t(\log(1/t))^{1+2\mu}},$$

where

$$\psi(t) = \log\left(\frac{1}{t}\right)\left[\alpha t^{2-\frac{1}{\gamma}}\lambda'(t) - (1-\alpha)\lambda t^{1-\frac{1}{\gamma}}\right] + (1+2\mu)\left[\left(1-\frac{\alpha}{\gamma}\right)\Lambda(t) + \alpha t^{1-\frac{1}{\gamma}}\lambda(t)\right].$$

As $\lambda(1) = 0$, we see that $\psi(1) = 0$. Thus, $g'(t) \le 0$ for $t \in (0, 1)$ is equivalent to showing that $\psi(t)$ is increasing on (0, 1). Now we compute

$$\psi'(t) = (1-\alpha)h(t) + t\alpha f(t), \tag{3.1}$$

where

$$h(t) = \left[\left(((1/\gamma) - 1)\log(1/t) - 2\mu\right)\lambda - t\lambda'(t)\log(1/t)\right] \tag{3.2}$$

and

$$f(t) = \left[\log(1/t)t\lambda''(t) + ((2-1/\gamma)\log(1/t) + 2\mu)\lambda'(t)\right]. \tag{3.3}$$

Thus, $\psi'(t) \ge 0$ for $t \in (0, 1)$ if and only if $h(t) \ge 0$ and $f(t) \ge 0$.

Now, we define ϕ by $\phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n$, where $b_n \ge 0$ for each $n \ge 1$, and consider

$$\lambda(t) = Kt^{B-1}(1-t)^{C-A-B}\phi(1-t), \tag{3.4}$$

where K is a constant satisfying the normalization condition that $\int_0^1 \lambda(t)dt = 1$. With the $\lambda(t)$ given by (3.4), by a simple computation, we find that

$$\lambda'(t) = Kt^{B-2}(1-t)^{C-A-B-1}[\big((B-1)(1-t) - (C-A-B)t\big)\phi(1-t) - t(1-t)\phi'(1-t)]$$

and

$$\begin{aligned}\lambda''(t) = {} & Kt^{B-3}(1-t)^{C-A-B-2}[((B-1)(B-2)(1-t)^2 \\ & -2(B-1)(C-A-B)t(1-t) + (C-A-B)(C-A-B-1)t^2)\phi(1-t) \\ & +[2(C-A-B)t - 2(B-1)(1-t)]t(1-t)\phi'(1-t) + t^2(1-t)^2\phi''(1-t)].\end{aligned}$$

If we substitute the values of $\lambda(t)$ and of $\lambda'(t)$ in (3.2), then, after simplifying the expression, we obtain

$$h(t) = X_1(t)\phi(1-t) + t(1-t)\log(1/t)\phi'(1-t), \tag{3.5}$$

where

$$X_1(t) = \left[\left(\frac{1}{\gamma} - 1\right)\log\left(\frac{1}{t}\right) - 2\mu\right](1-t) - \log\left(\tfrac{1}{t}\right)[(B-1)(1-t) - (C-A-B)t]. \tag{3.6}$$

Similarly by substituting the values of $\lambda'(t)$ and $\lambda''(t)$ in (3.3) we get

$$f(t) = \log(1/t)X(t) - 2\mu(1-t)Y(t)\phi(1-t) + Z(t)t(1-t)\phi'(1-t) + \log(1/t)t^2(1-t)^2\phi''(1-t)$$

with

$$\left.\begin{aligned} X(t) &= (1-B)\left(\frac{1}{\gamma} - B\right)(1-t)^2 + \left(\frac{1}{\gamma} - 2B\right)(C-A-B)t(1-t) + (C-A-B)(C-A-B-1)t^2 \\ Y(t) &= (1-B)(1-t) + (C-A-B)t \\ Z(t) &= \log(1/t)\left[2(C-A-B)t - \left(2B - \frac{1}{\gamma}\right)(1-t)\right] - 2\mu(1-t). \end{aligned}\right\} \tag{3.7}$$

The presence of the factor $\phi(1-t)$ in the last result is important in the sense that special cases give a number of interesting applications as we see in the following two theorems. Before we state these special results, it is necessary to introduce some notation. The well-known Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n \quad (|z| < 1),$$

where a, b and c are complex numbers with $c \neq 0, -1, -2, \ldots$. Here, for $a \neq 0$ and n a nonnegative integer, the Pochhammer symbol $(a)_n$ is given by $(a)_n = a(a+1)_{n-1}$ with $(a)_0 = 1$. A number of authors discussed the univalence, starlikeness and the convexity of the normalized hypergeometric functions $z\,{}_2F_1(a, b; c; z)$ and $(c/ab)\,[{}_2F_1(a, b; c; z) - 1]$, under a suitable restriction on the parameters a, b and c. For example, see [10–13].

Theorem 3.1. *Let $1/2 \leq \gamma \leq 1$, $\mu \in [0, 1/2]$ and $q_\gamma^\mu(t)$ be defined by (1.13). Suppose that β is given by*

$$\frac{\beta - 1/2}{1-\beta} = -(c+1)\int_0^1 t^c q_\gamma^\mu(t)dt,$$

where $c > -1$. Then for $f \in P_\gamma(\beta)$, the function

$$V_\lambda(f(z)) = (c+1)\int_0^1 t^c \frac{f(tz)}{t}dt$$

belongs to the class $M(\mu, \alpha)$ whenever c, $\alpha\gamma$ and μ satisfy the conditions

$$-1 < c \leq \min\{(1/\alpha) - 1, (1/\gamma) - 1\} \quad \text{and} \quad 0 < \alpha < 1. \tag{3.8}$$

Proof. By taking $\lambda(t) = (c+1)t^c$, we can see that

$$\lambda'(t) = c(c+1)t^{c-1}$$

and

$$\lambda''(t) = (c-1)c(c+1)t^{c-2}.$$

To prove the theorem, it is enough to show that (3.1) is nonnegative. Substituting $\lambda(t)$, $\lambda'(t)$ and $\lambda''(t)$ we can write (3.1) as,

$$\begin{aligned}
&(1-\alpha)\left[(((1/\gamma) - 1)\log(1/t) - 2\mu)\,(c+1)t^c - tc(c+1)t^{c-1}\log(1/t)\right] \\
&\quad + t\alpha\left[\log(1/t)t(c-1)c(c+1)t^{c-2} + ((2-1/\gamma)\log(1/t) + 2\mu)c(c+1)t^{c-1}\right] \\
&\quad = (1-\alpha)((1/\gamma - 1)\log(1/t) - 2\mu - c\log(1/t)) + \alpha\,(\log(1/t)c(c-1) + ((2-1/\gamma)\log(1/t) + 2\mu)c) \geq 0,
\end{aligned}$$

which upon simplification gives

$$\log(1/t)\,[(1/\gamma - 1 - c)(1 - \alpha - \alpha c)] - 2\mu(1 - \alpha - c\alpha) \geq 0. \tag{3.9}$$

For $\mu = 0$ the condition is true from the hypothesis, which is also obtained in [5]. Hence we consider the case $\mu \neq 0$. Then we can write (3.9) as

$$\log(1/t)\,[(c + 1 - 1/\gamma)(\alpha + \alpha c - 1)] + 2\mu(\alpha + c\alpha - 1) \geq 0$$

which is clearly positive for $c \leq \max\left\{\frac{1}{\alpha} - 1, \frac{1}{\gamma} - 1\right\}$. $\square$

Theorem 3.2. *Let $1/2 \leq \gamma \leq 1$, $\mu \in [0, 1/2]$ and $q_\gamma^\mu(t)$ be defined by (1.13). Suppose that β is given by*

$$\frac{\beta - 1/2}{1-\beta} = -K\int_0^1 t^{B-1}(1-t)^{C-A-B}\phi(1-t)q_\gamma^\mu(t)dt,$$

where A, B and $C > 0$, and K is a constant such that $K\int_0^1 t^{B-1}(1-t)^{C-A-B}\phi(1-t)dt = 1$. Then for $f \in P_\gamma(\beta)$, the function

$$V_\lambda(f(z)) = K\int_0^1 t^{B-1}(1-t)^{C-A-B}\phi(1-t)\frac{f(tz)}{t}dt$$

belongs to the class $M(\mu, \alpha)$ whenever A, B, C, γ and μ satisfy the conditions

$$0 < B \leq \frac{1}{2\gamma} - \mu \quad \text{and} \quad C \geq A + B + 1 + 2\mu. \tag{3.10}$$

Proof. To establish the theorem, it is enough to show that the inequality (3.1) is nonnegative under the stated conditions. Since $\phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n$ with $b_n \geq 0$ for all $n \geq 1$, each of the functions $\phi(1-t)$, $\phi'(1-t)$ and $\phi''(1-t)$ in the expression of $\psi'(t)$ are nonnegative for $t \in (0, 1)$. Therefore, to complete the proof, it suffices to show that the following inequalities hold for $t \in (0, 1)$: $X_1(t) \geq 0$, $Z(t) \geq 0$ and

$$\log(1/t)X(t) - 2\mu(1-t)Y(t) \geq 0, \tag{3.11}$$

where $X_1(t)$ and $X(t)$, $Y(t)$ and $Z(t)$ are defined by Eqs. (3.6) and (3.7) respectively. For convenience, we let $\delta = C - A - B$. The inequality $X_1(t) \geq 0$ implies

$$\left[\left(\frac{1}{\gamma} - 1\right)\log\left(\frac{1}{t}\right) - 2\mu\right](1-t) - \log\left(\frac{1}{t}\right)[(B-1)(1-t) - \delta t] \geq 0 \quad \text{or}$$
$$\left(\frac{1}{\gamma} - B\right)\log(1/t)(1-t) - 2\mu(1-t) + \delta t\log(1/t) \geq 0.$$

Using the fact $(\log(1/t))/(1-t) \geq 1$, we get

$$\frac{1}{\gamma} - B - \frac{2\mu}{1-t} + \frac{\delta t}{1-t} \geq 0 \quad \text{or} \quad \left(\frac{1}{\gamma} - B - 2\mu\right)(1-t) + (\delta - 2\mu)t \geq 0,$$

which is clearly true from (3.10).

Note that

$$2(1-t) \geq (1+t)\log(1/t) \tag{3.12}$$

holds for $t \in (0, 1)$. The truth of this inequality may be seen by comparing the coefficients of $(1-t)^n$ in the series expansion of both side functions in

$$\frac{1}{(1-(1-t)/2)} \leq -\frac{\log(1-(1-t))}{1-t}.$$

To prove (3.11), in view of (3.12), it is enough to show that $X(t) \geq \mu(1+t)Y(t)$. That is,

$$(1-B)\left(\frac{1}{\gamma} - B\right)(1-t)^2 + \left(\frac{1}{\gamma} - 2B\right)\delta t(1-t) + \delta(\delta-1)t^2 \geq \mu\left[(1-B)(1-t) + \delta t\right](1-t+3t),$$

which may be written as

$$(1-t)^2\left[(1-B)\left(\frac{1}{\gamma} - B - \mu\right)\right] + t(1-t)\left[\delta\left(\frac{1}{\gamma} - 2B - \mu\right) - 2\mu(1-B)\right] + \delta(\delta - 1 - 3\mu)t^2 \geq 0.$$

We claim that this inequality holds for all $t \in (0, 1)$. To do this, we observe that, under the conditions (3.10), both the coefficients of $(1-t)^2$ and t^2 in the left hand side expression of the last inequality given above are nonnegative. The coefficients of $t(1-t)$ are also nonnegative. Indeed

$$\begin{aligned}
\delta\left(\frac{1}{\gamma} - 2B - \mu\right) - 2\mu(1-B) &= (\delta - \mu)\left(\frac{1}{\gamma} - 2B - \mu\right) - \left[\mu^2 + \left(2 - \frac{1}{\gamma}\mu\right)\right] \\
&\geq (1+\mu)\left(\frac{1}{\gamma} - 2B - \mu\right) - \left[\mu^2 + \left(2 - \frac{1}{\gamma}\mu\right)\right] \\
&= (1+\mu)\left[\frac{1}{\gamma} - 2B - 2\mu + \left(\frac{1}{\gamma} - 1\right)\frac{\mu}{1+\mu}\right] \geq 0.
\end{aligned}$$

Therefore, (3.11) holds for $t \in (0, 1)$. It remains to verify the inequality $Z(t) \geq 0$ for $t \in (0, 1)$. Again, in view of (3.12), the inequality $Z(t) \geq 0$ holds if

$$2\delta t - \left(2B - \frac{1}{\gamma}\right)(1-t) \geq \mu(1+t)$$

is true for $t \in (0, 1)$. We may rewrite this inequality as

$$[2(\delta - 2\mu) + \mu]t + 2\left(\frac{1}{2\gamma} - B - \mu\right)(1+t)\mu \geq 0,$$

which is clearly true under the condition (3.10). Thus, we obtain $\psi(t) \geq 0$ under the stated conditions and hence, we complete the proof by applying Theorem 2.1. □

Theorem 3.3. *Let $a, b, c > 0$, $1/2 \leq \gamma \leq 1$, $\mu \in [0, 1/2]$ and $q_\gamma^\mu(t)$ be defined by (1.13). Suppose that β is given by*

$$\frac{\beta - 1/2}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b-1)}\int_0^1 t^{b-1}(1-t)^{c-a-b}F\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; z\right) q_\gamma^\mu(t)dt.$$

Then for $f \in P_\gamma(\beta)$, the function $zF(a, b; c; z) * f(z)$ belongs to $M(\alpha, \mu)$ whenever a, b, c, γ and μ satisfy the conditions

$$0 < a \leq 1,\ 0 < b \leq \frac{1}{2\gamma} - \mu \quad \textit{and} \quad c \geq a + b + 1 + 2\mu. \tag{3.13}$$

Proof. By choosing

$$\phi(1-t) = F\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right)$$

we can write

$$\lambda(t) = Kt^{b-1}(1-t)^{c-a-b}\phi(1-t),$$

where

$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b-1)}.$$

Now substituting A, B and C in Theorem 3.2 by appropriate values of a, b and c, respectively the conclusion follows. □

Theorem 3.4. *Let $1/2 \le \gamma \le 1$, $\mu \in [0, 1/2]$ and $q_\gamma^\mu(t)$ be defined by (1.13). Suppose that β is given by*

$$\frac{\beta - 1/2}{1-\beta} = -\frac{(1+c)^p}{\Gamma(p)}\int_0^1 t^c \left(\log\frac{1}{\gamma}\right)^{p-1} q_\gamma^\mu(t)dt.$$

Then for $f \in P_\gamma(\beta)$, the function

$$V_\lambda(f) = \frac{(c+1)^p}{\Gamma(p)}\int_0^1 \left(\log\frac{1}{\gamma}\right)^{p-1} t^{c-1} f(tz)dt,$$

belongs to $M(\alpha, \mu)$ whenever a, b, c, γ and μ satisfy the conditions

$$p \ge 2(1+\mu) \quad \text{and} \quad 0 \le c+1 \le (1/2\gamma) - \mu.$$

Proof. Choose $\phi(1-t) = \left(\frac{\log(1/t)}{1-t}\right)^{p-1}$. If we take $C - A - B = p - 1$ and $B = c + 1$ then $\lambda(t)$ takes the form

$$\lambda(t) = Kt^c(1-t)^{p-1}\phi(1-t), \quad K = \frac{(1+c)^p}{\Gamma(p)}.$$

We complete the proof by applying Theorem 3.2. □

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